Universidade do Minho
Escola de Engenharia

Noise Resilient Quantum Amplitude Estimation

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Work supervised by:

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INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

INESCTEC

FCT Fundação para a Ciência e a Tecnologia
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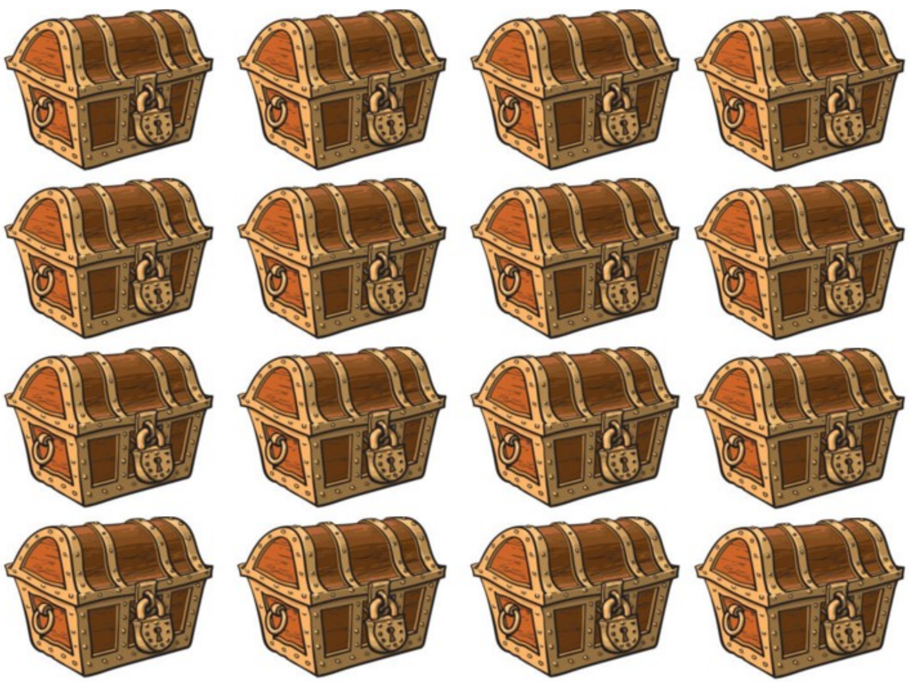
Overview

1. Quantum searching
2. Quantum amplitude estimation
3. Noisy quantum devices
4. Numerical results
5. Overview and numerical analysis of QAE algoritms
6. Bayesian amplitude estimation

Quantum Searching



Quantum resources can bring a **quadratic speed up** to the task of **searching an unstructured**

Quantum Searching

Mathematically:

$$f(x) : \{0,1\}^n \rightarrow \{0,1\}$$

$$f(x) = \begin{cases} 1 \text{ , if } x \text{ is a solution;} \\ 0 \text{ , otherwise.} \end{cases}$$

 ; 

find x s. t. $f(x) = 1$

Quantum Searching

$$f(x) = \begin{cases} 1 \text{ , if } x \text{ is a solution;} \\ 0 \text{ , otherwise.} \end{cases}$$

Classical approach:

1. Sample uniformly at random.
2. If , terminate.
3. Go to 1.

Cost:

$$\sum_{k=1}^{N} \frac{1}{N} \cdot k = \frac{1}{N} \cdot \left(\frac{N(N+1)}{2} \right) = \frac{N+1}{2}$$

$\in \quad \mathcal{O}(N)$

or if there are >1 solutions:

$$\mathcal{O}(N/M)$$

Quantum Searching

Assume we can encode as a quantum phase oracle .

$$f(x) = \begin{cases} 1 \text{ , if } x \text{ is a solution;} \\ 0 \text{ , otherwise.} \end{cases}$$

$$\hat{U}_f \mid x \rangle = (-1)^{f(x)} \mid x \rangle$$

Quantum Searching

ntum approach:

Prepare a superposition $| \psi_A \rangle = A | 0 \rangle^{\otimes n}$ containing the solution.

Quantum Searching

ntum approach:

repare a superposition $|\psi_A\rangle = A|0\rangle^{\otimes n}$ containing the solution.

E.g. use the Hadamard transform: $|\psi_A\rangle = H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{N}}\sum_{x\in\{0,1\}^n}|x\rangle$

Call the subset of solutions, its complementary in . Define:

$$|\Psi_1\rangle = \sqrt{\frac{1}{M}}\sum_{x\in X}|x\rangle \quad ; \quad |\Psi_0\rangle = \sqrt{\frac{1}{N-M}}\sum_{x\in X^c}|x\rangle$$

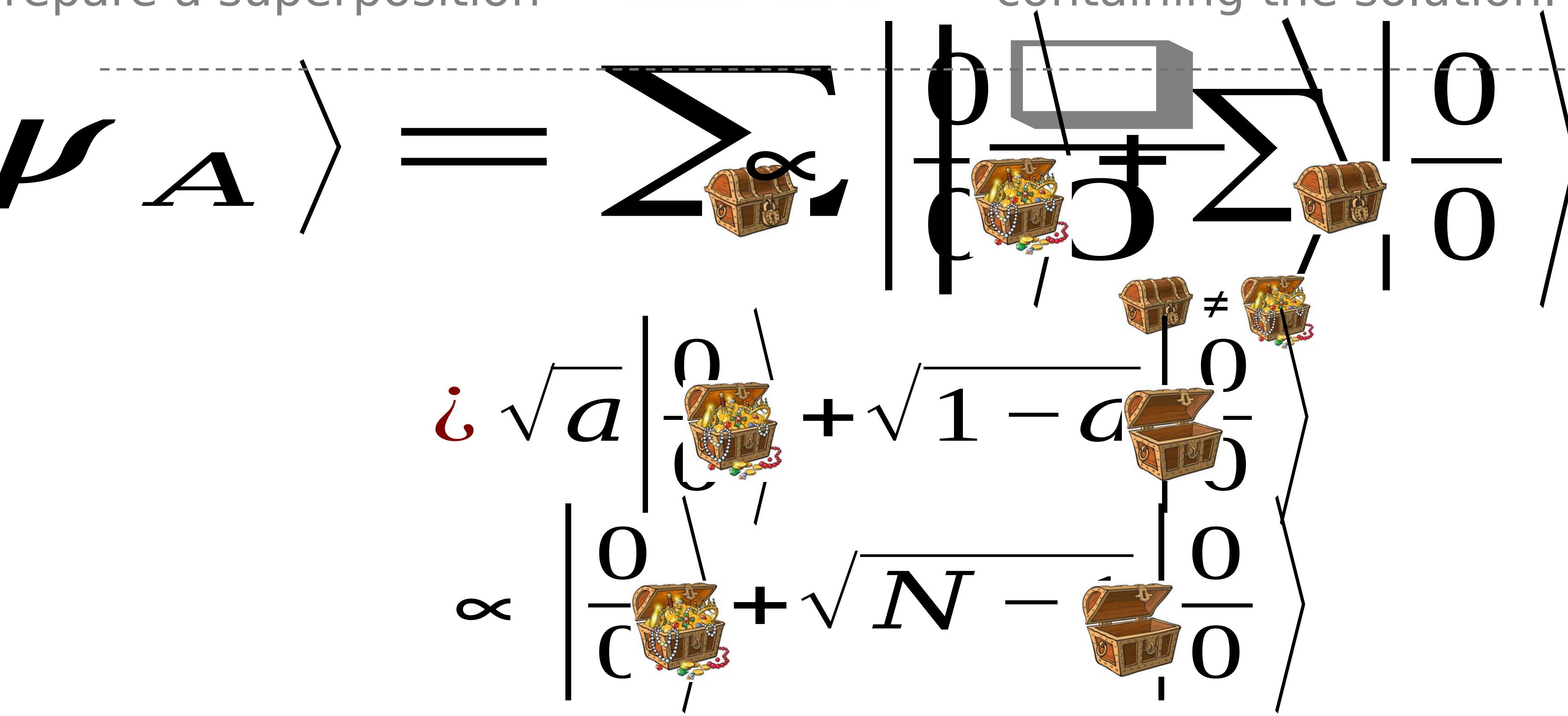
Then
:

$$|\psi_A\rangle = \sqrt{a}\,|\psi_1\rangle + \sqrt{1-a}\,|\psi_0\rangle \qquad a \equiv \frac{M}{N}$$

Quantum Searching

ntum approach:

Prepare a superposition $|\psi_A\rangle = A|0\rangle^{\otimes n}$ containing the solution.



Quantum Searching

ntum approach:

Prepare a superposition $| \psi_A \rangle = A | 0 \rangle^{\otimes n}$ containing the solution.

Amplify the pre-image of 1 under via quantum amplitude amplification (

Quantum Searching

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Amplify the pre-image of 1 under via quantum amplitude amplification (

Defin
e $\theta = \arcsin(\sqrt{a}) \leftrightarrow a = \sin^2(\theta)$.

Then $|\psi_A\rangle = \sqrt{a}\,|\psi_1\rangle + \sqrt{1-a}\,|\psi_0\rangle$
,
becomes
:

$$|\psi_A\rangle = \sin(\theta)\,|\psi_1\rangle + \cos(\theta)\,|\psi_0\rangle$$

$$\hat{G} = -A\,\hat{U}_0\,A^{-1}\,\hat{U}_f$$

$$\hat{G}^m\,|\psi_A\rangle = \sin((2m+1)\theta)\,|\psi_1\rangle + \cos((2m+1)\theta)\,|\psi_0\rangle$$

$$|\langle\psi_1|\hat{G}^m|\psi\rangle|^2 = \sin^2((2m+1)\theta)$$

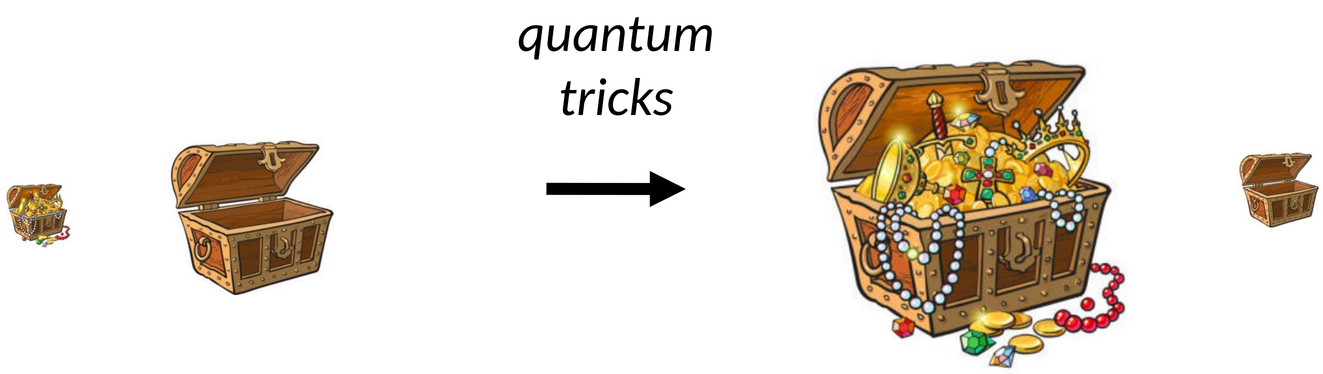
$$(2m_{\text{ideal}}+1)\theta = \pi/2$$

Quantum Searching

ntum approach:

Prepare a superposition $|\psi_A\rangle = A|0\rangle^{\otimes n}$ containing the solution.

Amplify the pre-image of 1 under via quantum amplitude amplification (



Quantum Searching

ntum approach:

Prepare a superposition $|\psi_A\rangle = A|0\rangle^{\otimes n}$ containing the solution.

Amplify the pre-image of 1 under via quantum amplitude amplification (

$|\psi\rangle \left|\frac{0}{0}\right.$  $\rangle + \sqrt{N-1}$  $\left|\frac{0}{0}\right\rangle$

QAA

$|\psi\rangle = \sqrt{1\,a_{\text{amp}}}\,|0$  $\rangle + \sqrt{1-a_{\text{an}}}$  $\left|\frac{0}{0}\right\rangle$

Quantum Searching

ntum approach:

Prepare a superposition $|\psi_A\rangle = A|0\rangle^{\otimes n}$ containing the solution.

Amplify the pre-image of 1 under via quantum amplitude amplification (

Measure to find a solution with high probability.

Quantum Searching

ntum approach:

repare a superposition $|\psi_A\rangle = A|0\rangle^{\otimes n}$ containing the solution.

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leasure to find a solution with high probability.

Measure

$$|\psi\rangle = \sqrt{1\,a_{\text{amp}}}\,|0\rangle + \sqrt{1 - a_{\text{an}}}\,\left|\frac{0}{0}\right\rangle$$

to obtain  with probability.

Quantum Searching

ntum approach:

repare a **superposition** $|\psi_A\rangle = A|0\rangle^{\otimes n}$ containing the solution.

mplify the pre-image of 1 under via quantum amplitude amplification

leasure to find a solution with high probability.

$$|\psi\rangle \propto |0\rangle + \sqrt{N-1}\,|0\rangle$$


↓ *amplify*

$$|\psi\rangle = \sqrt{1 a_{\text{amp}}}\,|0\rangle + \sqrt{1 - a_{\text{am}}}\,|0\rangle$$


for large

↓ *measure*

observe $|0\rangle$ with high probability

Quantum Searching

$|\psi\rangle \left|\frac{0}{0}\right\rangle$  $+\sqrt{N-}$  $\left|\frac{0}{0}\right\rangle$

↓ *amplify*

$|\psi\rangle = \sqrt{1\,a_{\text{amp}}}\left|\frac{0}{0}\right\rangle$  $+\sqrt{1-a_{\text{am}}}$  $\left|\frac{0}{0}\right\rangle$ *for large*

↓ *measure*

observe $\left|\frac{0}{0}\right\rangle$  with high probability

Cost: $\Theta(\sqrt{1/a})$ $\Theta(\sqrt{N/M})$

Quantum Amplitude Estimation

Quantum amplitude estimation (QAE) is related to quantum searching; it occurs in the same framework, and offers the same **quadratic speed-up**. It consists on the task of estimating the parameter :

$$|\psi\rangle = \sqrt{a}\,|0\rangle + \sqrt{1-a}\,|0\rangle$$

$$a = ?$$

Quantum Amplitude Estimation

$$|\psi\rangle = \sqrt{a}\,|0\rangle + \sqrt{1-a}\,|0\rangle$$

$$a = ?$$

It becomes relevant when the problem is generalized to consider more than one distinguished item, or even fractional amounts thereof – so that ***a* can be any real number in the unit interval**.

Quantum Amplitude Estimation

$$|\psi\rangle = \sqrt{a}\,|0\rangle + \sqrt{1-a}\,|0\rangle$$

$$a = ?$$

Classical approach: sum over samples.

$$\frac{1}{N}\sum_{x} f(x) = \frac{1}{N}\sum_{x\in X} 1 = \frac{1}{N}M = a$$

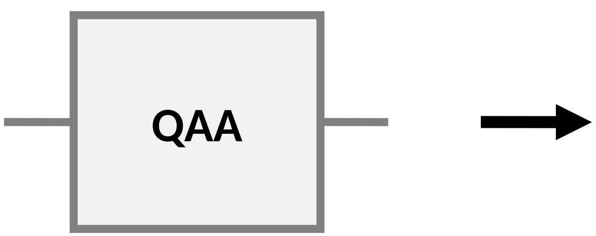
$$\epsilon \in O(1/\sqrt{K})$$

Quantum Amplitude Estimation

Quantum approach: perform **phase estimation on the amplification oper** $\lambda_{\pm} = \exp(\pm i2\theta)$

eigenvalues are directly

related to *a*.



Quantum Phase Estimation

We measure one of two eigenphases: or

$\epsilon \in O(1/K)$

Quantum Amplitude Estimation

$|\psi\rangle = \sqrt{a}\,|0$  $\rangle + \sqrt{1-a}\,|0$  $\rangle$

$$a = ?$$

This estimation task is a fundamental routine, with a wide range of applications in **chemistry**, **machine learning** and **statistics**.

Quantum Amplitude Estimation

In particular, it can be used to **speed-up Monte**

$$\mathbb{E}_{p(x)}[f(x)] = \int_{\Omega} f(x)p(x)\mathrm{d}x$$

$$\mathbb{E}_{p(x)}[f(x)] \approx \frac{1}{N}\sum_{i=0}^{N-1}\frac{f(x_i)\cdot p(x_i)}{\pi(x_i)} \qquad \{x_i\}_{i=0}^{N-1} \sim \pi(\cdot)$$

$$¿\sum_{x} p(x) f(x)$$ for a uniform PMF

Quantum Amplitude Estimation

Define **distribution loading** and **function encoding**

$$P\,|\,0\,\rangle^{\otimes n} = \sum_{x=0}^{2^n-1} \sqrt{p(x)}\,|\,x\,\rangle \qquad R\,|\,x\,\rangle\,|\,0\,\rangle = |\,x\,\rangle\left(\sqrt{f(x)}\,|\,1\,\rangle + \sqrt{1-f(x)}\,|\,0\,\rangle\right)$$

$$R(P \otimes I_1)\,|\,0\,\rangle^{\otimes n} = \sum_{x=0}^{2^n-1} \sqrt{p(x)}\,|\,x\,\rangle\left(\sqrt{f(x)}\,|\,1\,\rangle + \sqrt{1-f(x)}\,|\,0\,\rangle\right)$$

Rewrite in terms of orthonormal subspaces and normalize to get:

$$|\,\psi\,\rangle = |\,\psi_1\,\rangle + |\,\psi_0\,\rangle = \sqrt{a}\left|\,\tilde{\psi}_1\right\rangle + \sqrt{1-a}\left|\,\tilde{\psi}_0\right\rangle$$

$$a \equiv \sum_x p(x) f(x) = \mathbb{E}_{p(x)}[f(x)]$$

We have an amplitude estimation problem!

Quantum Amplitude Estimation

Define **distribution loading** and **function encoding**

$$P \mid 0 \rangle^{\otimes n} = \sum_{x=0}^{2^n-1} \sqrt{p(x)} \mid x \rangle \qquad R \mid x \rangle \mid 0 \rangle = \mid x \rangle \left(\sqrt{f(x)} \mid 1 \rangle + \sqrt{1 - f(x)} \mid 0 \rangle \right)$$

$$R(P \otimes I_1) \mid 0 \rangle^{\otimes n} = \sum_{x=0}^{2^n-1} \sqrt{p(x)} \mid x \rangle \left(\sqrt{f(x)} \mid 1 \rangle + \sqrt{1 - f(x)} \mid 0 \rangle \right)$$

Rewrite in terms of orthonormal subspaces:

$$\mid \psi_1 \rangle = \sum_{x=0}^{2^n-1} \sqrt{p(x)} \sqrt{f(x)} \mid x \rangle \mid 1 \rangle \qquad \mid \psi_0 \rangle = \sum_{x=0}^{2^n-1} \sqrt{p(x)} \sqrt{1 - f(x)} \mid x \rangle \mid 0 \rangle$$

Normalize to get:

$$\mid \psi \rangle = \mid \psi_1 \rangle + \mid \psi_0 \rangle = \sqrt{a} \left| \tilde{\psi}_1 \right\rangle + \sqrt{1 - a} \left| \tilde{\psi}_0 \right\rangle$$

We have an amplitude estimation problem!

Quantum Amplitude Estimation

$$a \equiv \sum_x p(x) f(x) \approx \mathbb{E}_{p(x)}[f(x)]$$

Our **initialization** and **oracle** operators are now:

$$A \equiv R(P \otimes I_1) \qquad \hat{U}_j = (I_n \otimes Z)$$

Example: integrate $\frac{1}{b_{\max}} \int_0^{b_{\max}} \sin^2(x) \mathrm{d}x$ using a uniform importance distribution.

$$R \,|\, x \,\rangle \,|\, 0 \,\rangle = |\, x \,\rangle \left(\sin\left(\frac{(x + \frac{1}{2}) b_{\max}}{2^n} \right) |\, 1 \,\rangle + \cos\left(\frac{(x + \frac{1}{2}) b_{\max}}{2^n} \right) |\, 0 \,\rangle \right) \longrightarrow R = \prod_{k=1}^{n} \mathrm{C}^{(k)} R_y \left(\frac{b_{\max} \cdot x_k}{2^{n-k}} \right) \left(I_n \otimes R_y \left(\frac{b_{\max}}{2^n} \right) \right)$$

$$= \frac{b_{\max}}{2^n}/2 + \sum_{k=1}^{n} \frac{b_{\max} \cdot x_k}{2^{n-k}}/2$$

$$P = H^{\otimes n}$$

Quantum Amplitude Estimation

e can calculate the measurement probability for any bit strin

$$P(\text{measuring } x \mid QAE(\theta)) = \frac{P(\text{measuring } x \mid QPE(\theta/\pi)) + P(\text{measuring } x \mid QPE(1-\theta/\pi))}{2}.$$

with

$$P(\text{measuring } x \mid QPE(\phi)) = \frac{\sin^2(K\Delta\pi)}{K^2 \sin^2(\Delta\pi)},$$

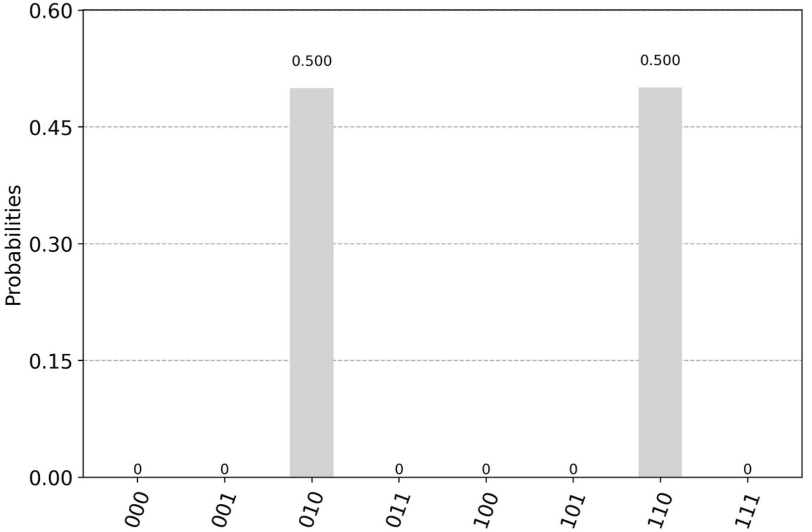
where Δ is a circular distance, and also the error in the estimate produced by x:

$$\Delta = \left|\phi - \frac{x}{K}\right| \mod 1.$$

Quantum Amplitude Estimation

When is an integer, we retrieve the exact solution by measuring.

Example bar plot (deterministic case):



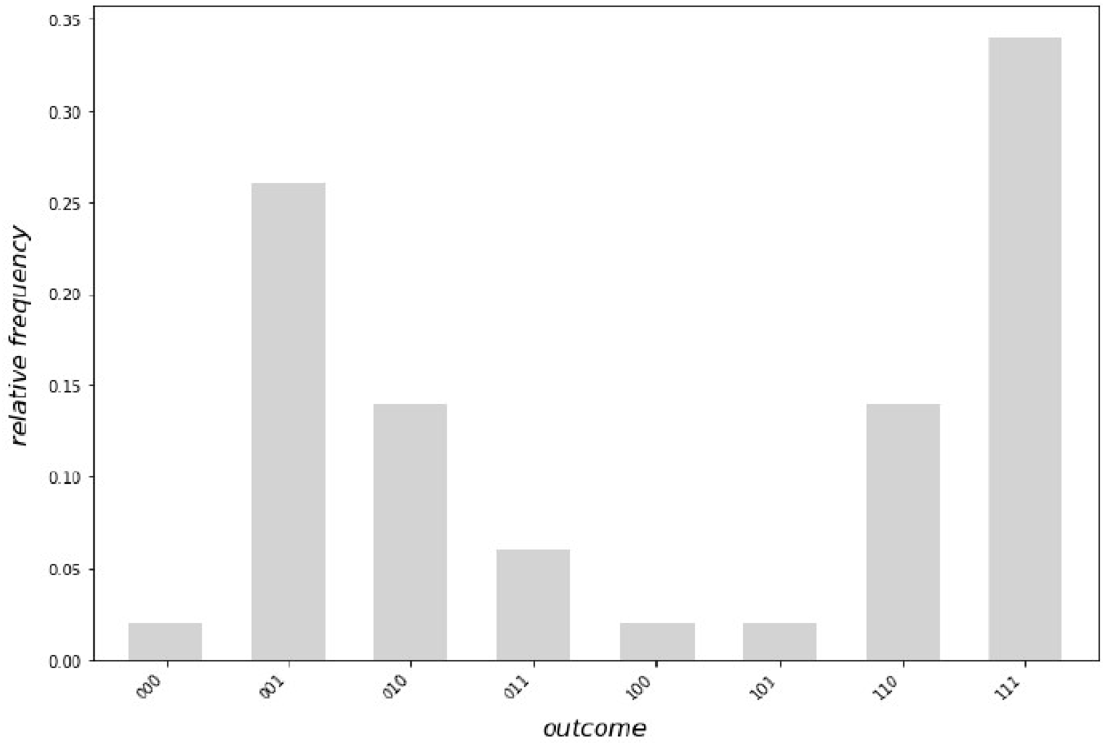
- 50 measurements

$$
\left.\begin{aligned}
x &= 0b010 = 2 \rightarrow \theta = \frac{x\pi}{K} = \frac{2\pi}{8} \\
x &= 0b110 = 6 \rightarrow \theta = \frac{x\pi}{K} = \frac{6\pi}{8}
\end{aligned}\right\}
$$

Quantum Amplitude Estimation

When is not an integer, the outcome distribution is not so neat.

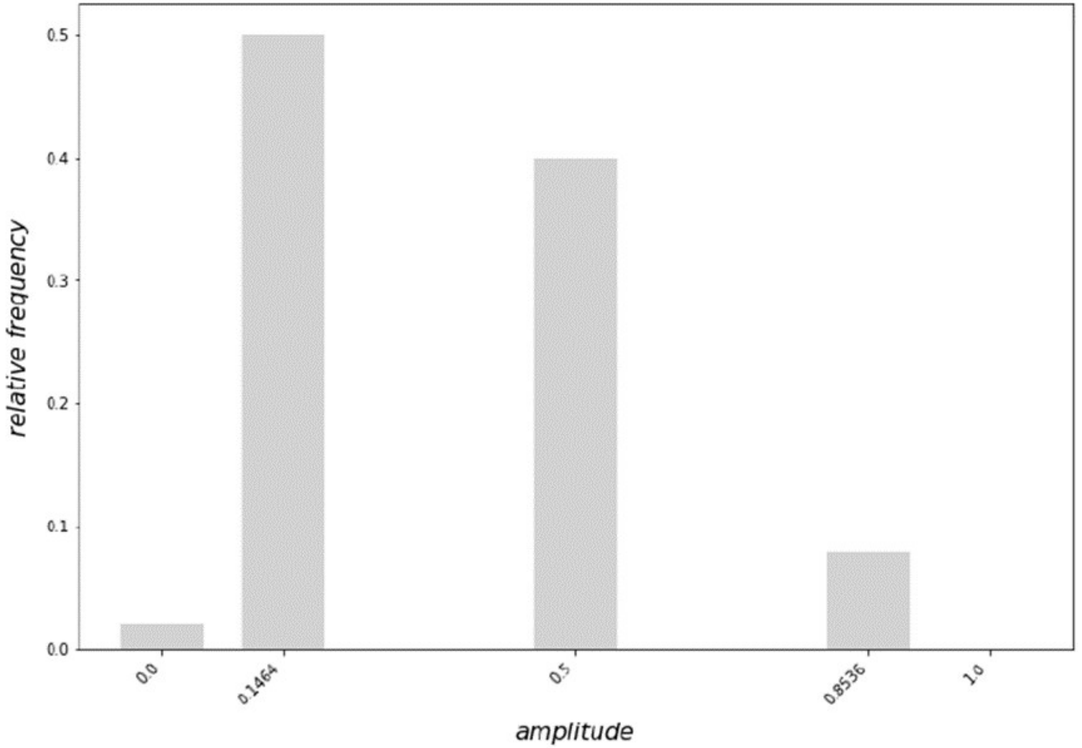
Example bar plot:



50 measurements

Quantum Amplitude Estimation

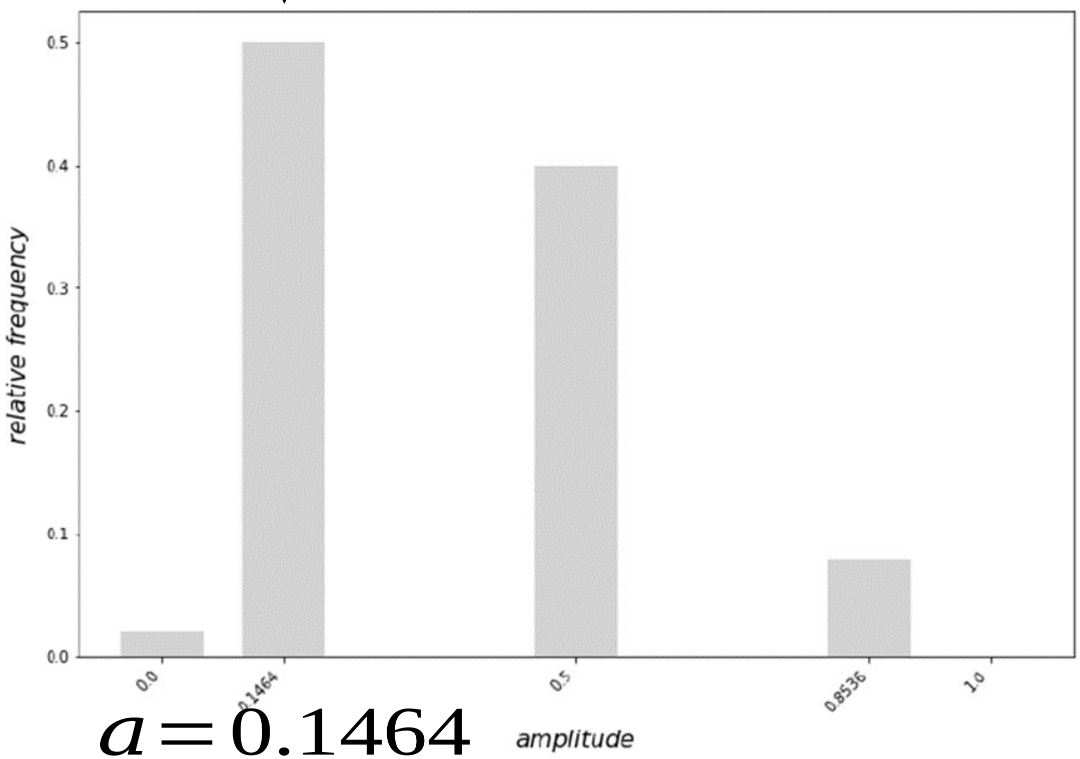
We can still translate it into the amplitude domain:



- 50 measurements

Quantum Amplitude Estimation

closest binary fraction

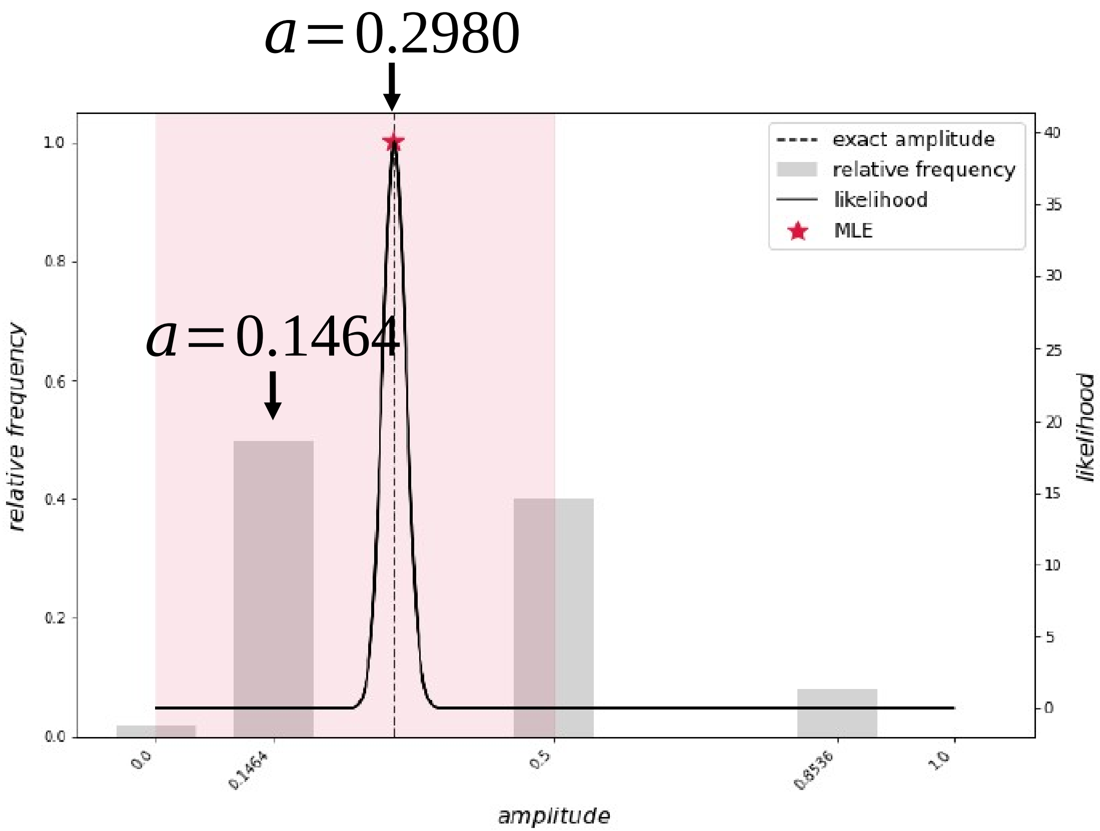


$a = 0.1464$

- 50 measurements

Quantum Amplitude Estimation

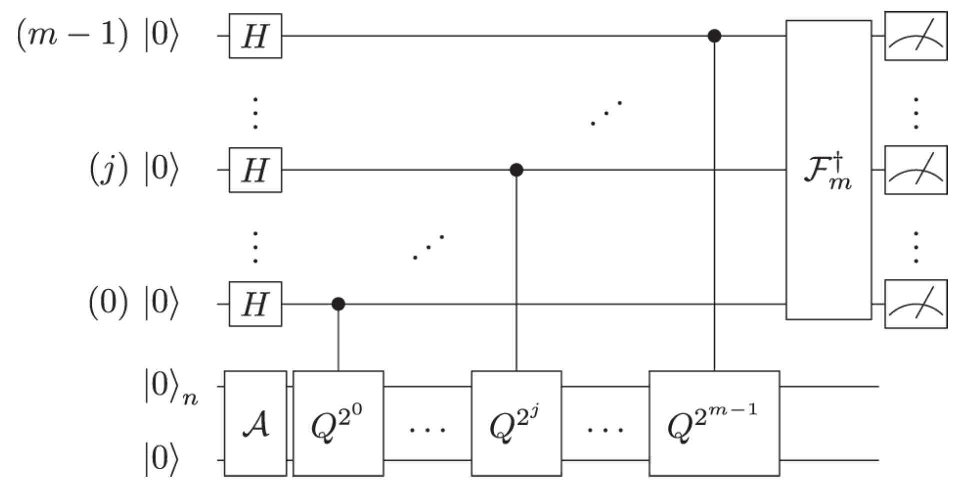
Rather than sticking to a grid, we can use our knowledge of the outcome distribution to sweep over a continuum of values for .



- 50 measurements

Quantum Amplitude Estimation

Today's quantum devices are **unable** to gainfully realize **QPE**.

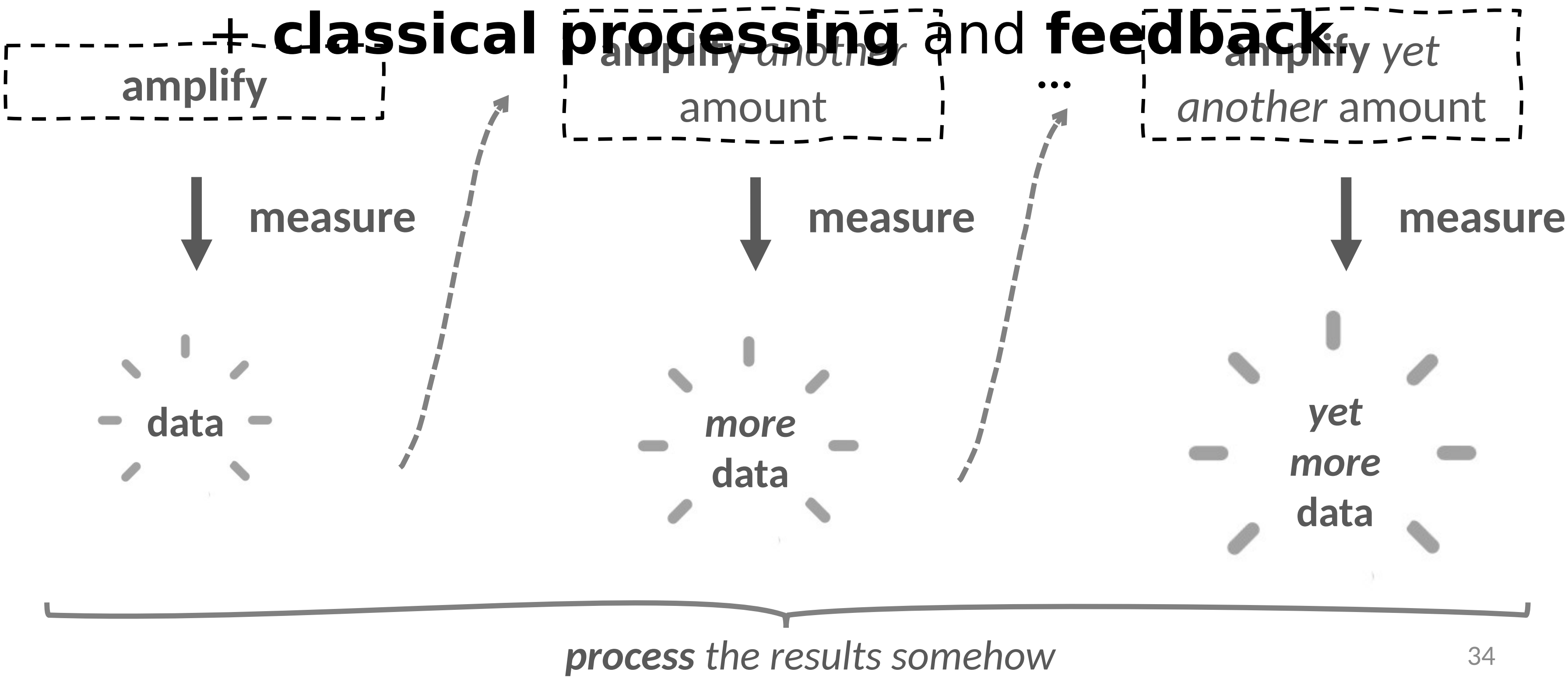


in: Grinko, D., Gacon, J., Zoufal, C. et al. Iterative quantum amplitude estimation. npj Quantum Inf ***7****, 52 (2021).*

As such, **alternative strategies** have been proposed to achieve **quantum-enhanced precision** in more **hardware-friendly** ways.

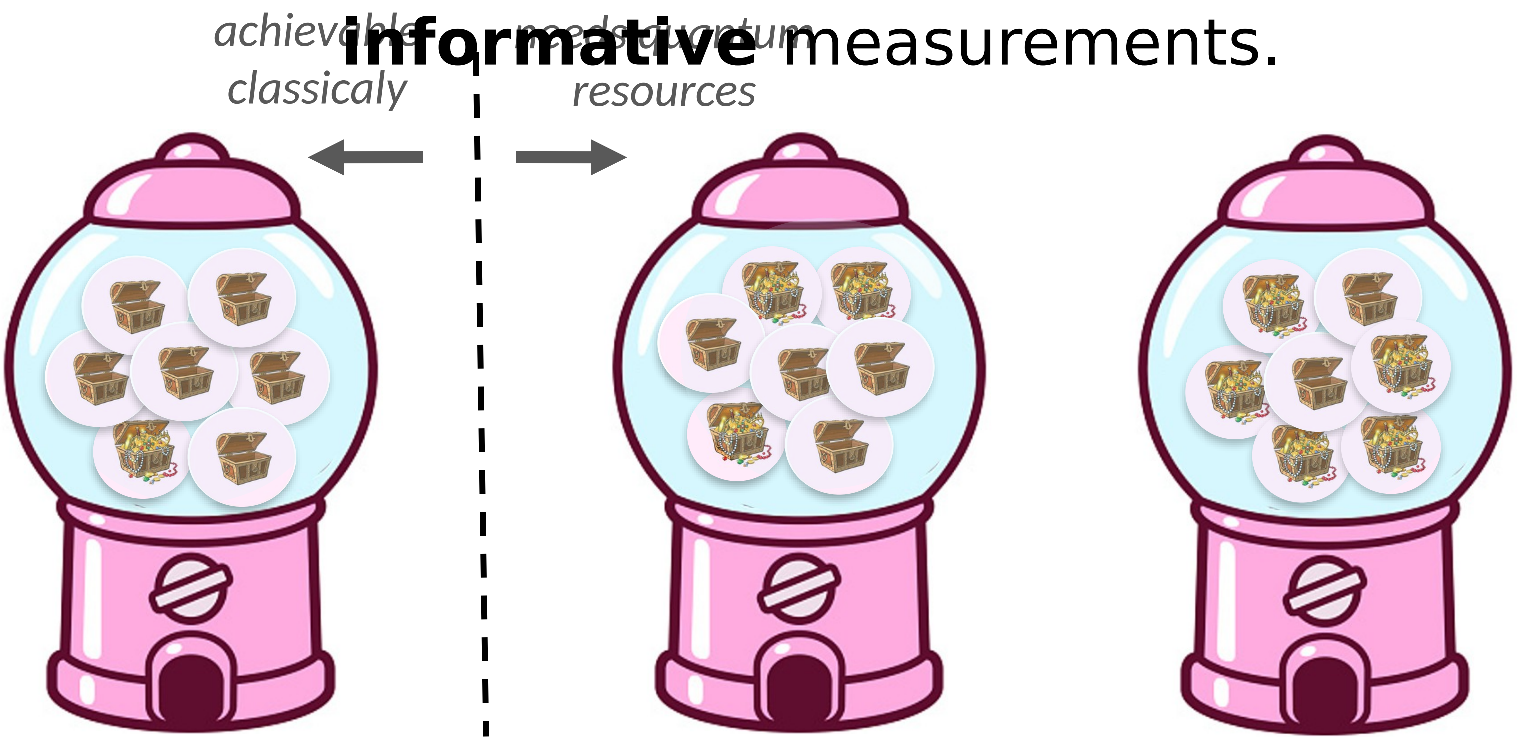
Quantum Amplitude Estimation

These are approaches are often **hybrid** and **adaptive**, relying on **simpler quantum circuits + classical processing** and **feedback**.



Quantum Amplitude Estimation

The overarching idea is to use **quantum amplitude amplification** to make **more informative** measurements.



Quantum Amplitude Estimation

$$Binomial\left(p\left(\theta\right), N_{shots}\right)$$

Classically, we can sample from:

Quantum-enhanced measurements allow:

for any odd integer

Quantum Amplitude Estimation

With ,

Preparing   entails queries to A (forwards or backwards):

- for the initialization of
- for the applications of

It also requires queries to the oracle, one for each application of .

We can sample according to the probability

using a number of queries is in .

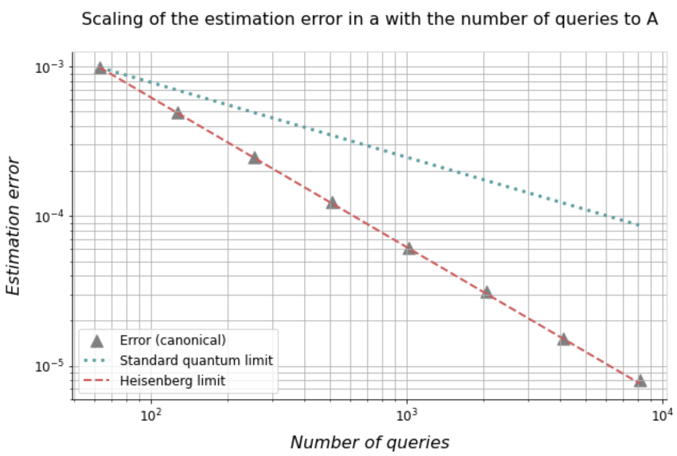
How to use these measurements is an open question.

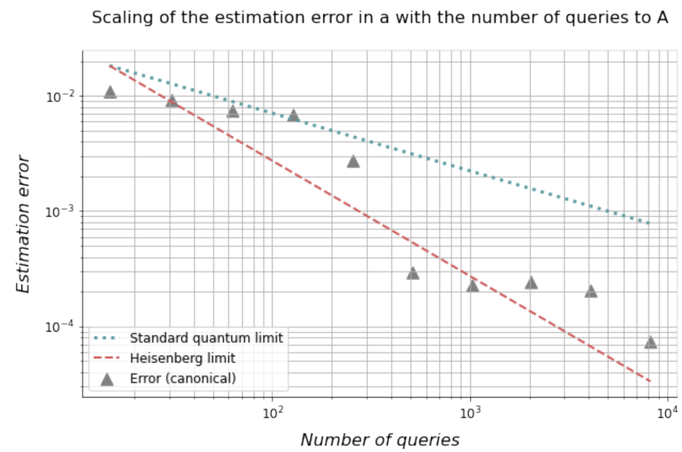
Each answer brings a different **quantum advantage**,

classical overhead and **noise resilience**.

Overview and numerical analysis of QAE algoritms

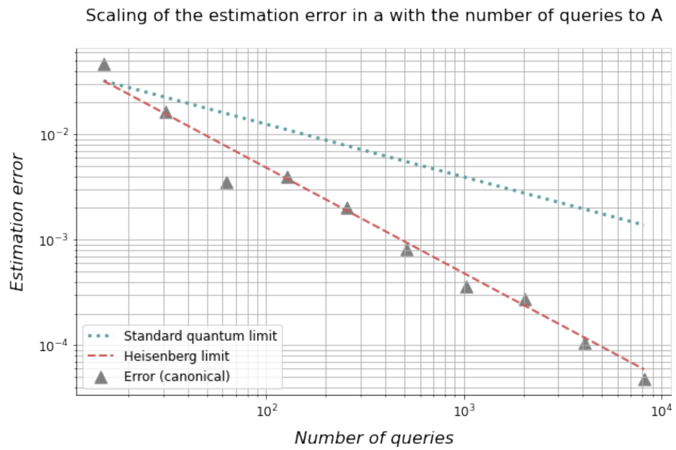
Textbook QAE





$a = 0.5$

$a = \frac{\pi}{10}$

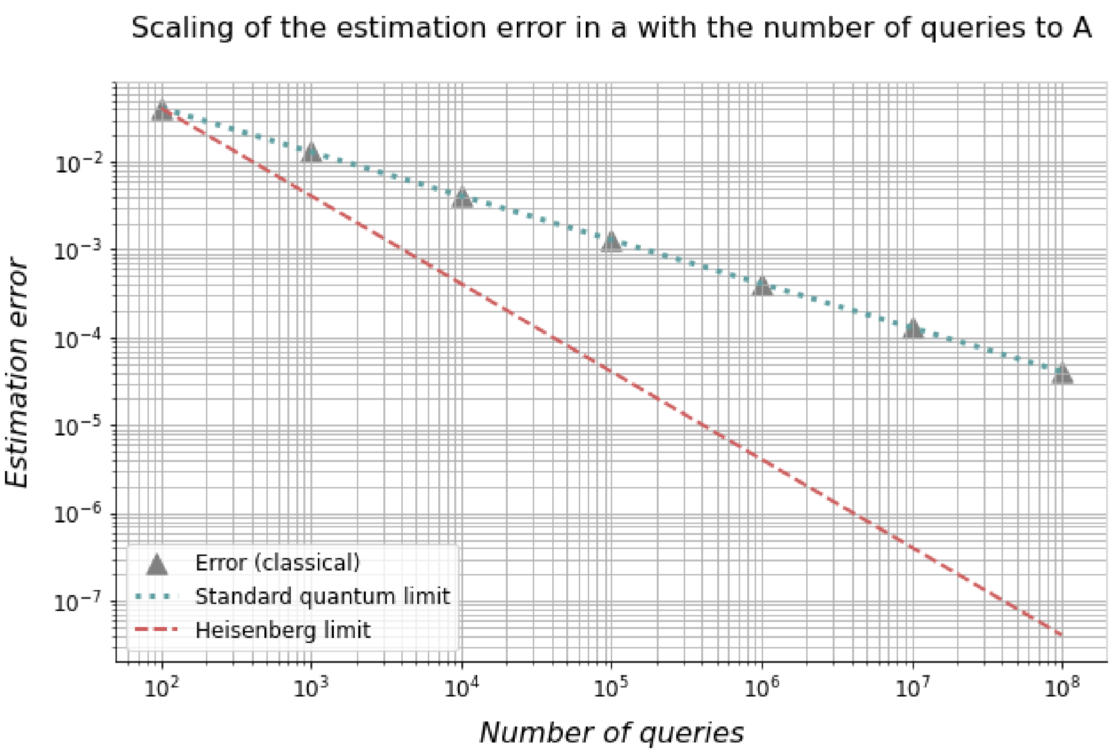


$$\epsilon^{(HL)} \propto \frac{1}{N}$$

a Unif¿

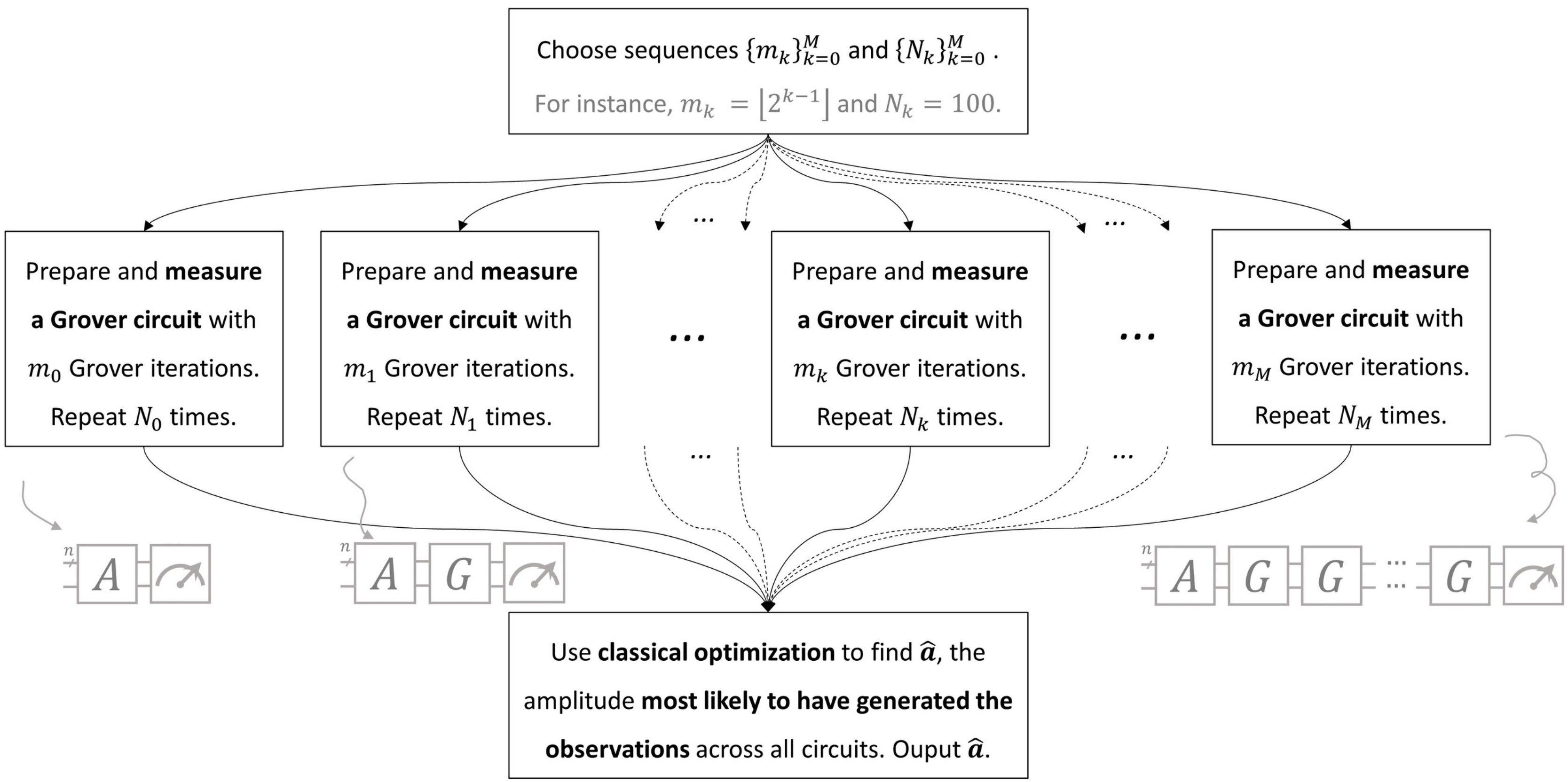
Classical Amplitude Estimation

Compare with the classical case:



$$\epsilon^{(SQL)} \propto \frac{1}{\sqrt{N}}$$

Maximum Likelihod QAE

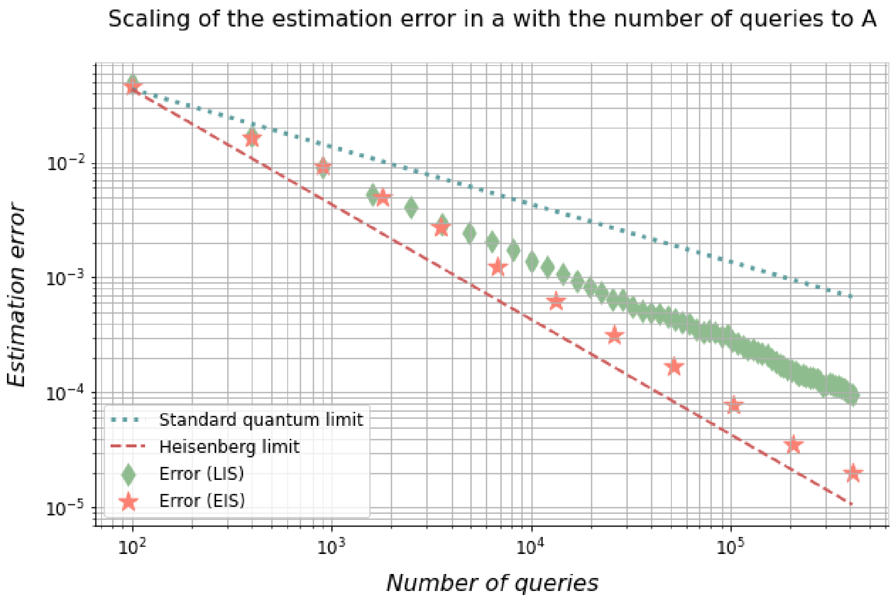


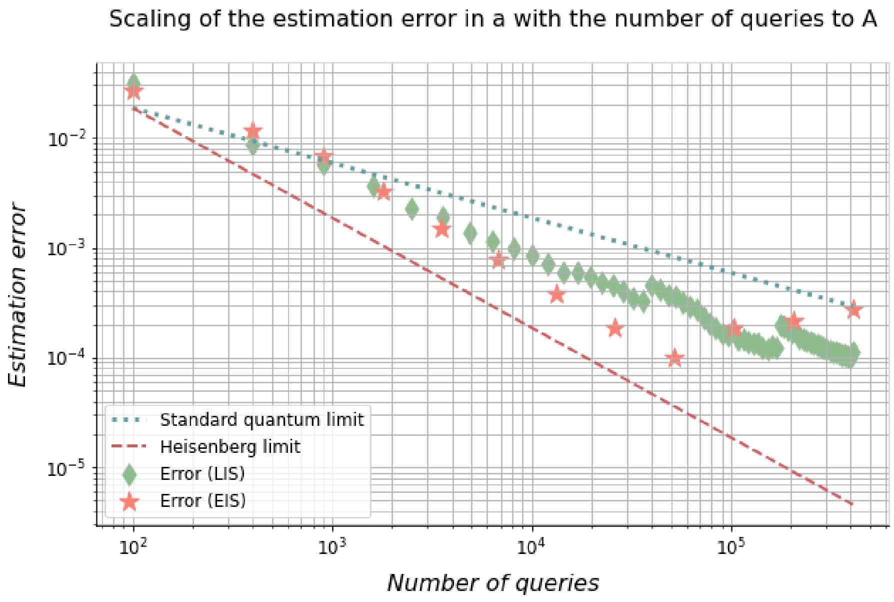
By Suzuki et al, 2020.

Maximum Likelihod QAE

- Original proposal (Suzuki et al): ;
- Spin offs:
  - Variational QAE
  - Low depth QAE $N \in \mathcal{O}(1/\epsilon^{1+\beta})$ and $D \in \mathcal{O}(1/\epsilon^{1-\beta})$
    - Power Law: , a controllable parameter
    - QoPrime

Maximum Likelihod QAE

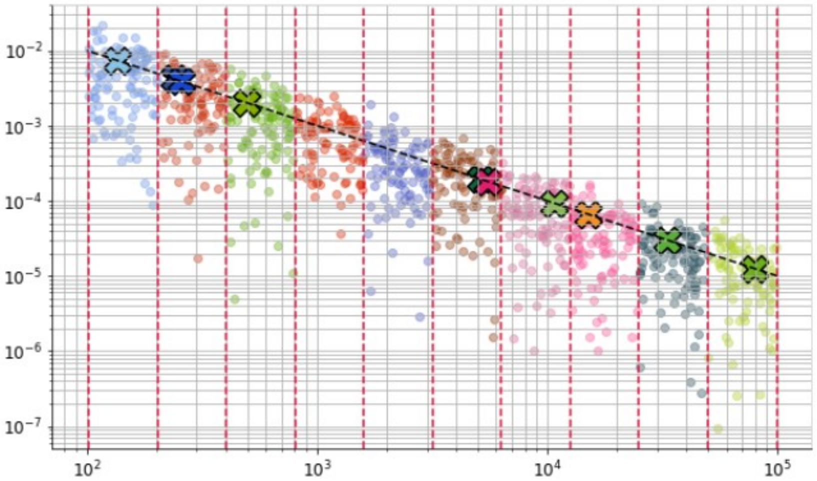




$a = 0.5$,

Quantum Amplitude Estimation

A note on performance benchmarking for adaptive algorithms



- Sample an x (representing N) coordinate uniformly at random on a log scale.

$$x \sim \text{unif}_{\text{log}}([x_{\text{min}}, x_{\text{max}}])$$

- Sample auxiliary variables z depending on x:

$$z \sim \mathcal{N}(\mu, \sigma(x)),$$

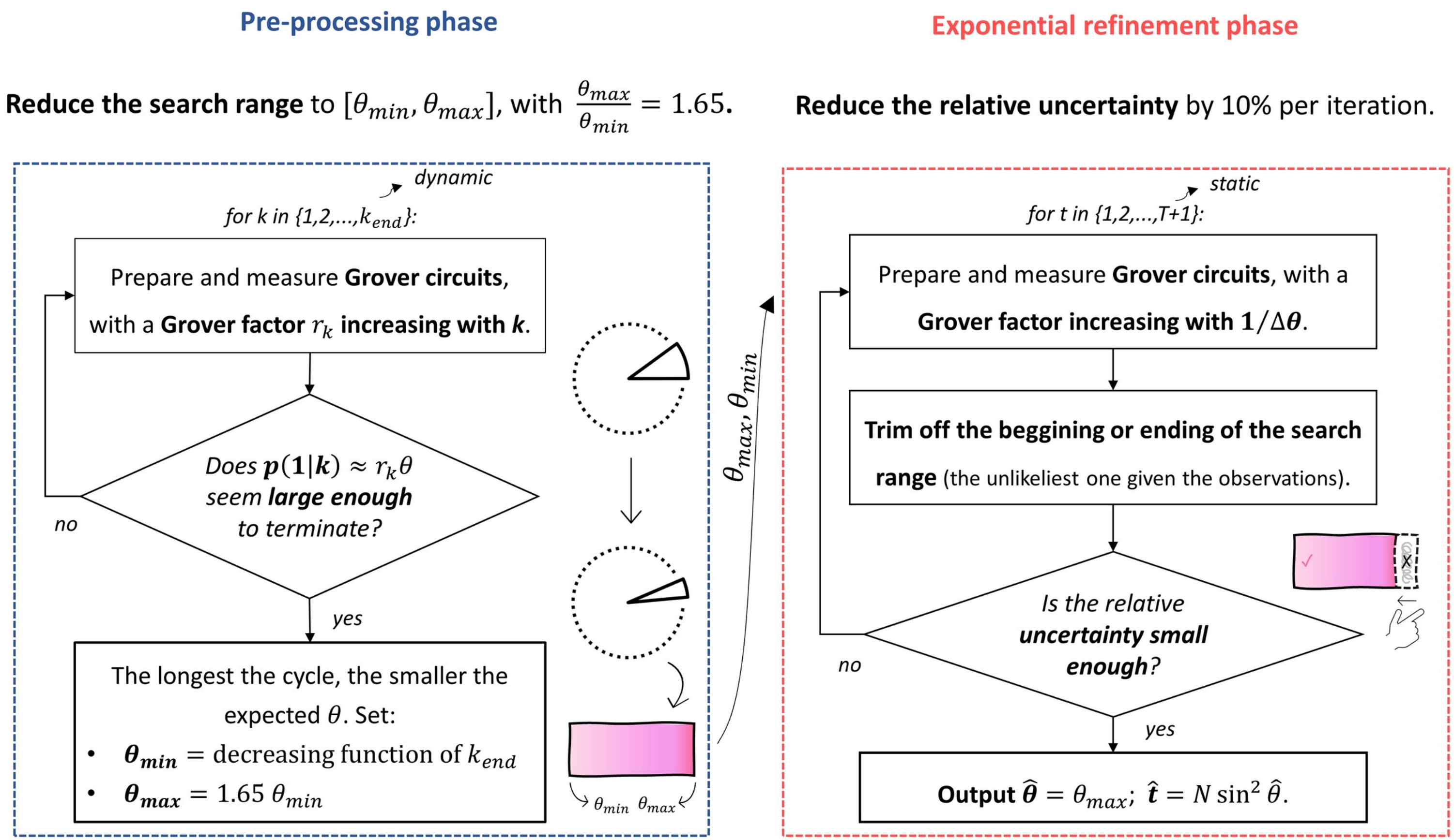
where

$$\sigma(x) \propto 1/x.$$

The choice of the mean μ is arbitrary, whereas the constant of proportionality is determined by fixing a point.

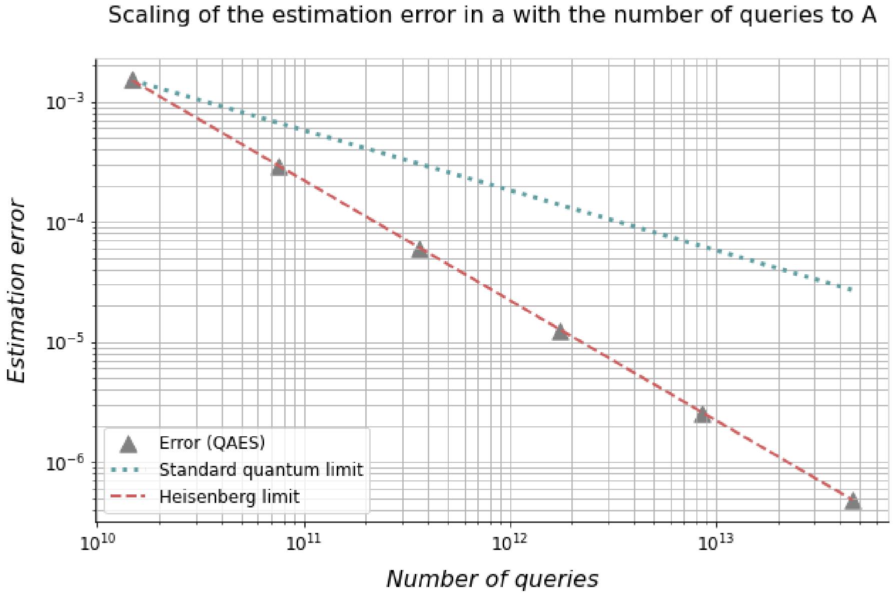
- Calculate y (representing ϵ^2) as $y = (x - \mu)^2$.

QAE, simplified



By Aaronson et al, 2020

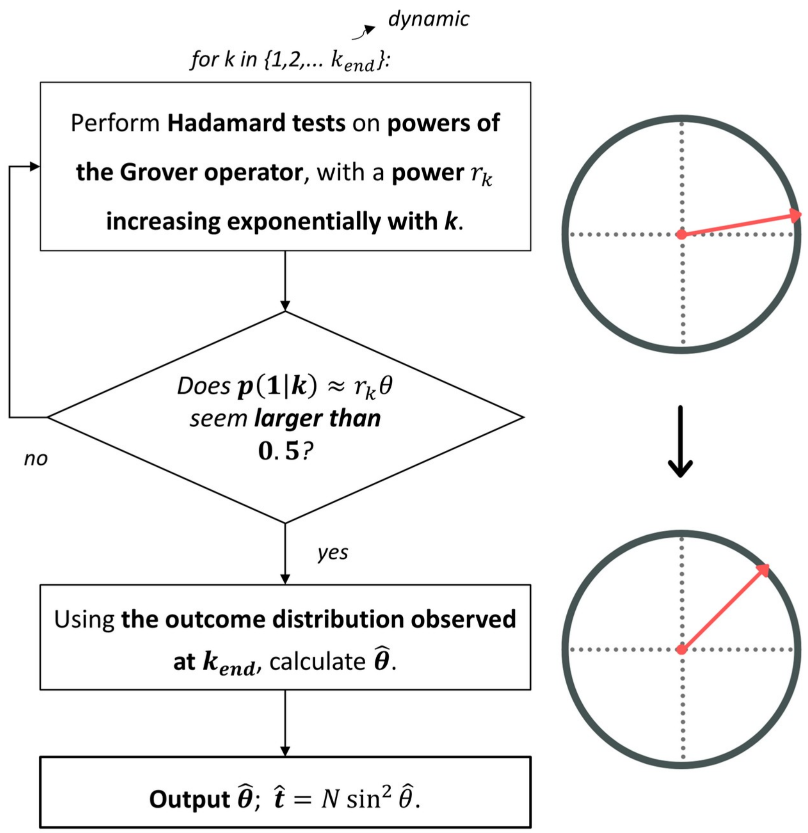
QAE, simplified



$a = 0.5$

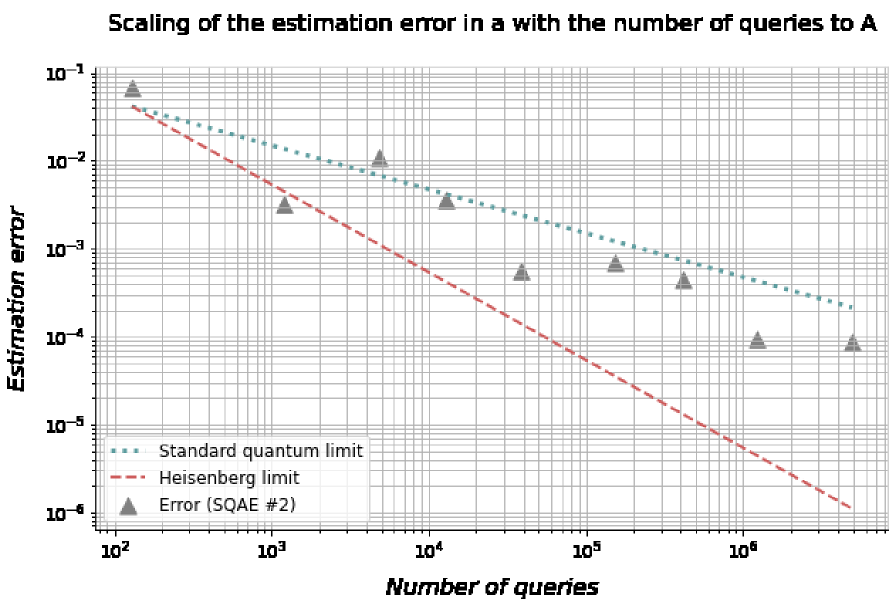
,

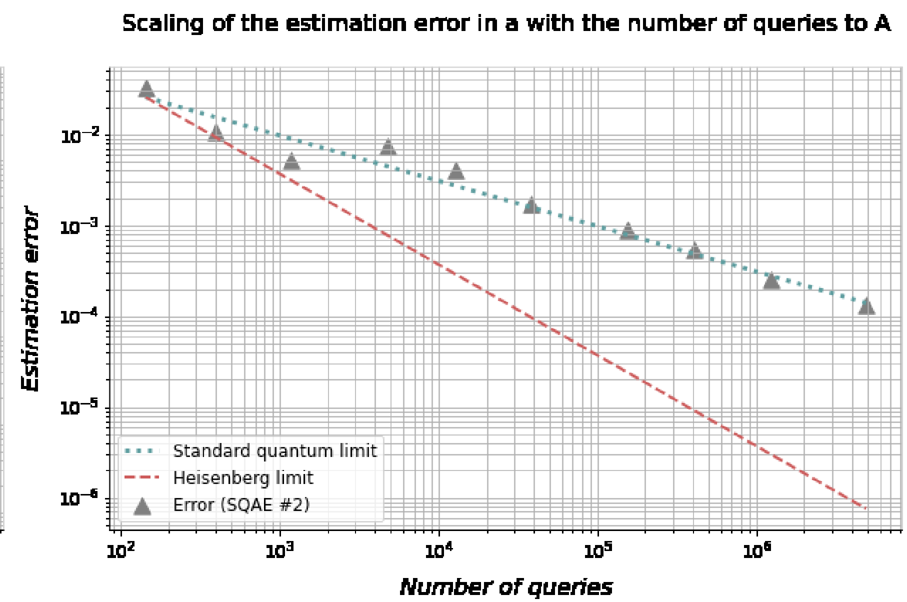
Simpler QAE



By Wie, 2020

Simpler QAE





$a = 0.5$

,

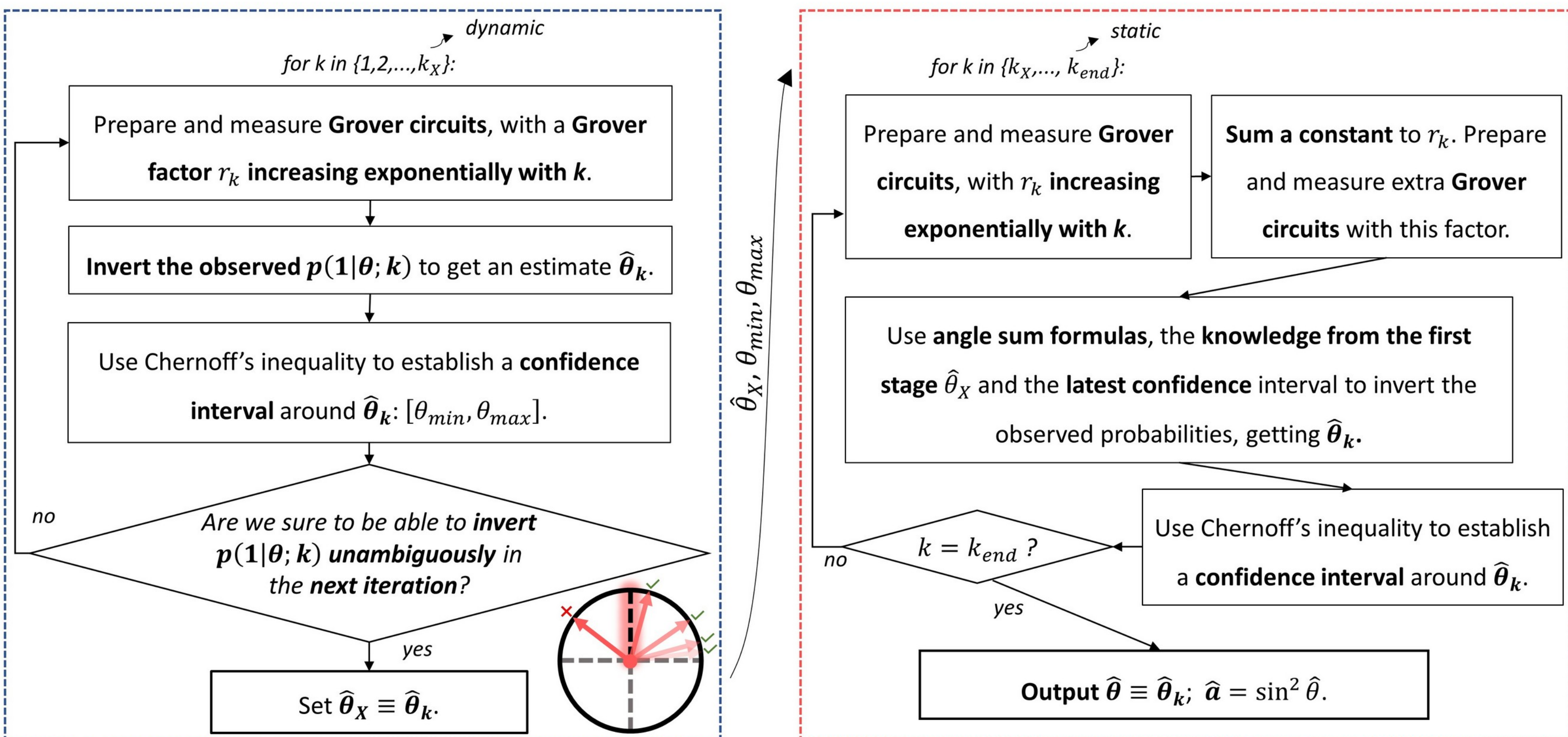
Faster QAE

First stage

Learn by **inverting** the results of **Grover measurements** with **increasing amplification,** until they **become ambiguous**.

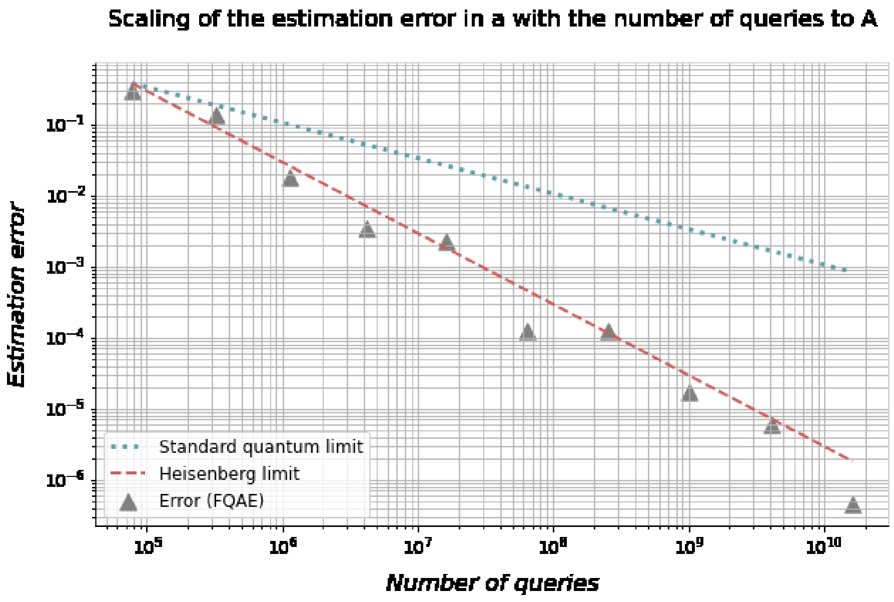
Second stage

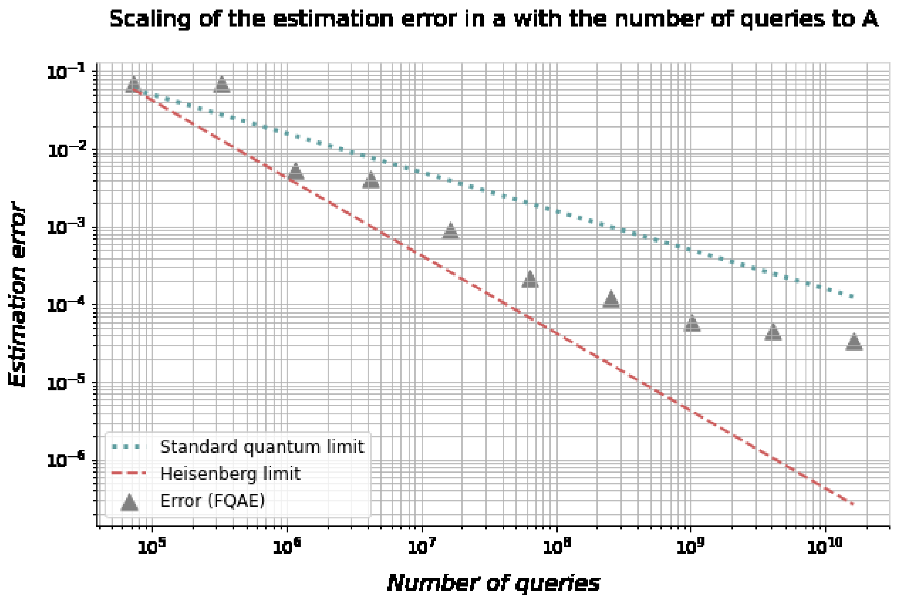
Continue to learn by **complementing the former Grover measurements** with additional ones for **disambiguation**.



By Nakaji, 2020

Faster QAE





$a = 0.5$

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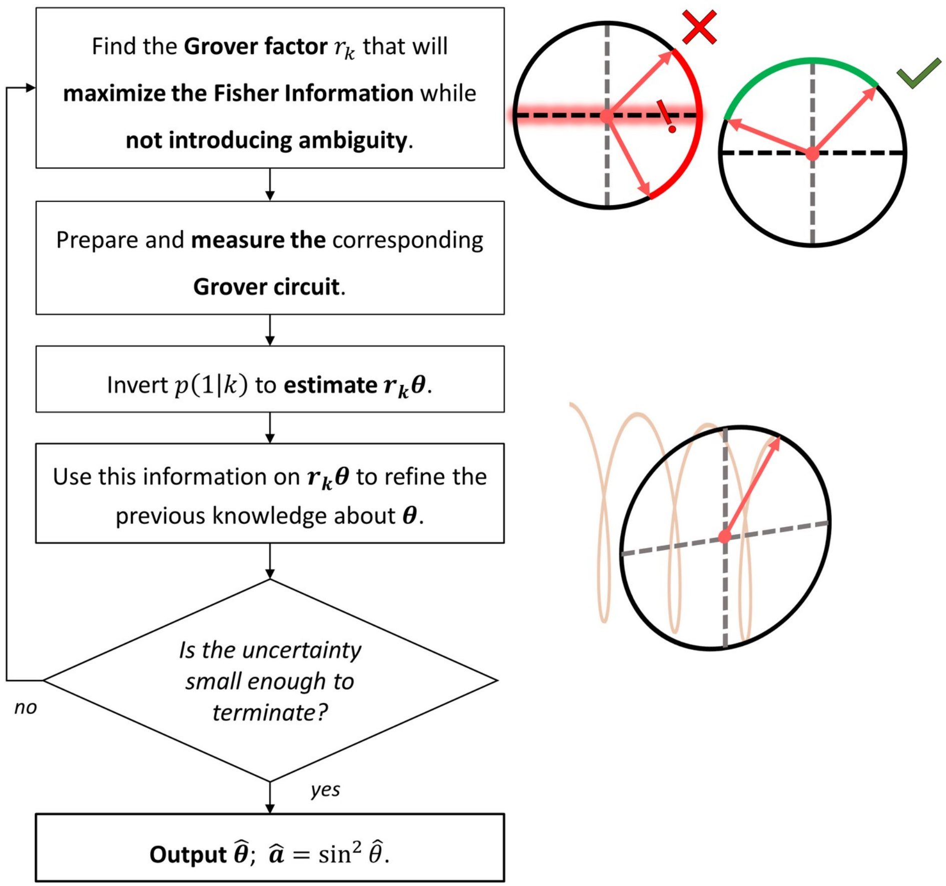
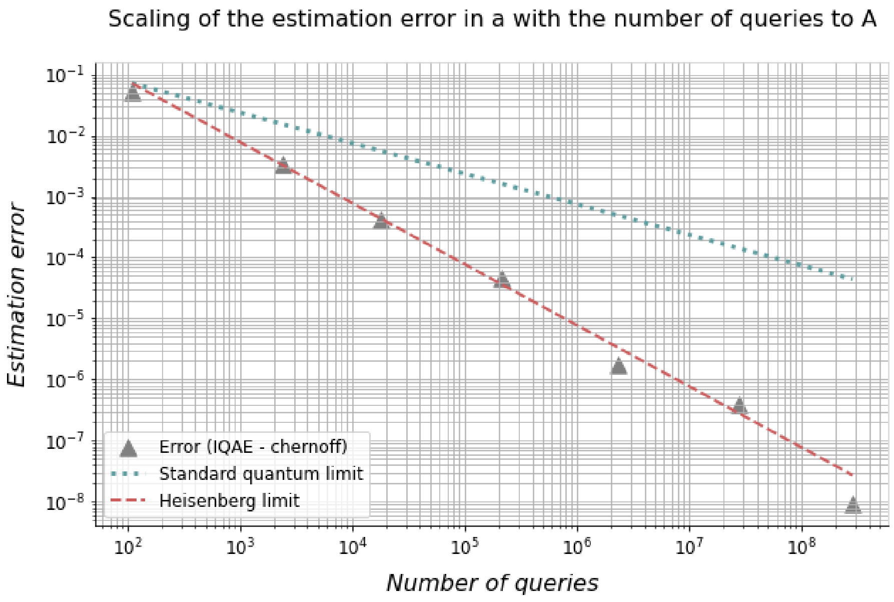
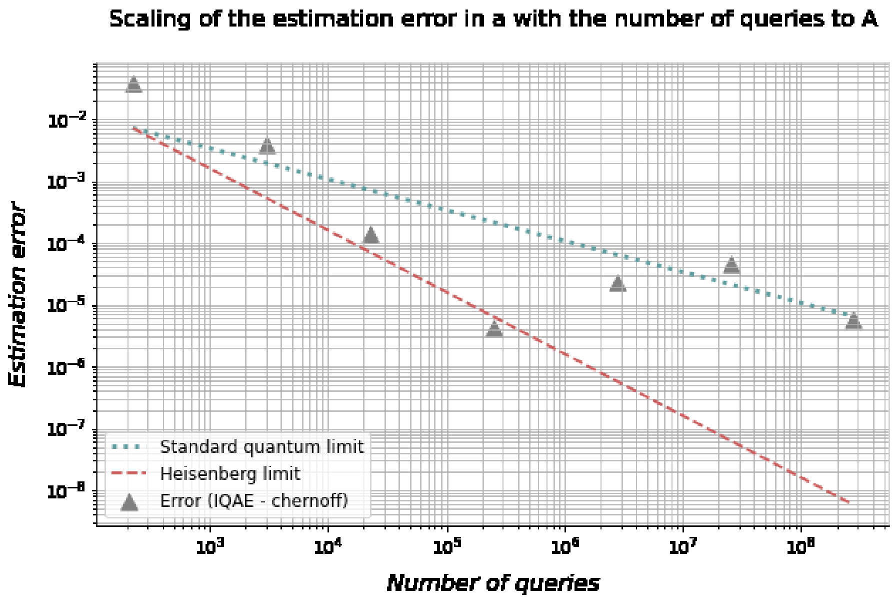
Iterative QAE



By Grinko et al, 2021. Modified version by Fukuzawa et al, 2022

Iterative QAE





$a = 0.5$

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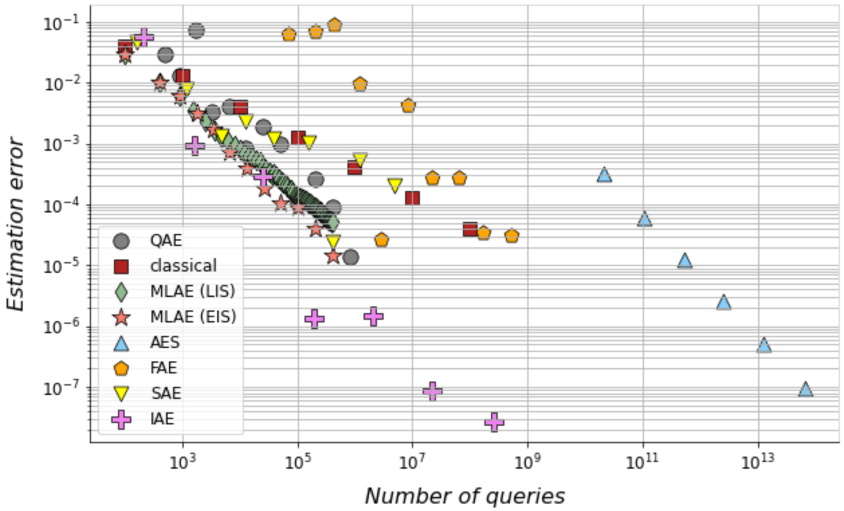
Summary

	Key idea	Parallelizable	Circuits	Main strength(s)	Main weak-ness(es)	Complexity
QAE [4]	QPE on Grover operator	no	QPE	optimal complexity	very deep circuit; noise oblivious	$N_q \in O(\frac{1}{\epsilon}\log\frac{1}{\alpha}\cdot a^{-1})$
MLAE [56]	heuristic measure-ments → statistical estimation	fully	QAA	simplicity; solid numerical performance	no formal guarantees; unchecked circuit growth	best case | observed: $N_q^{(LIS)} \sim \epsilon^{-0.75} \mid \epsilon^{-0.76}$ $N_q^{(EIS)} \sim \epsilon^{-1} \mid \epsilon^{-0.88}$
V-MLAE [47]	MLAE with variational approximation	fully	QAA	solid numerical performance; limits circuit depth	no formal guarantees; cost overhead	similar to MLAE except for EIS-observed (untested)
AES [1]	rough localization → exponential refinement	partly	QAA	optimal complexity	large cost offset; noise oblivious	$N_q \in O(\frac{1}{\epsilon}\log\frac{1}{\alpha}\cdot a^{-1})$
SAE [61]	amplify → invert probability	partly	Hadamard tests	–	inconclusive demonstrations	$N_q \in O(a^{-1})$
IAE [22]	watchful choice of Fisher information	partly	QAA	nearly optimal complexity	unchecked circuit growth; noise oblivious	$N_q \in O(\frac{1}{\epsilon}\log(\frac{1}{\alpha}\log\frac{1}{\epsilon}))$
M-IAE [17]	IAE but dis-tribute shots more favorably	partly	QAA	optimal complexity	same as IAE	$N_q \in O(\frac{1}{\epsilon}\log\frac{1}{\alpha})$
FAE [42]	complementary measurements → invert probability	partly	QAA	nearly optimal complexity	unchecked circuit growth; noise oblivious	$N_q \in O(\frac{1}{\epsilon}\log(\frac{1}{\alpha}\log\frac{1}{\epsilon})$

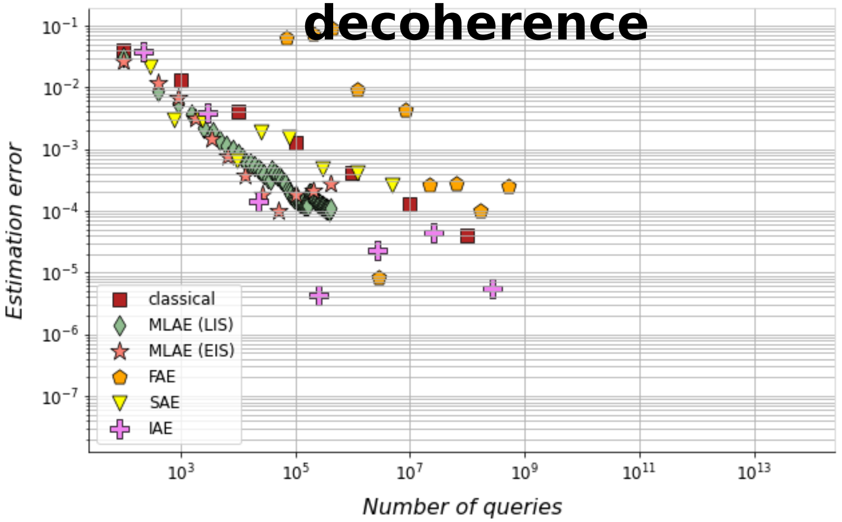
Summary

ideal performance



$a = 0.1$

performance in the presence of **decoherence**



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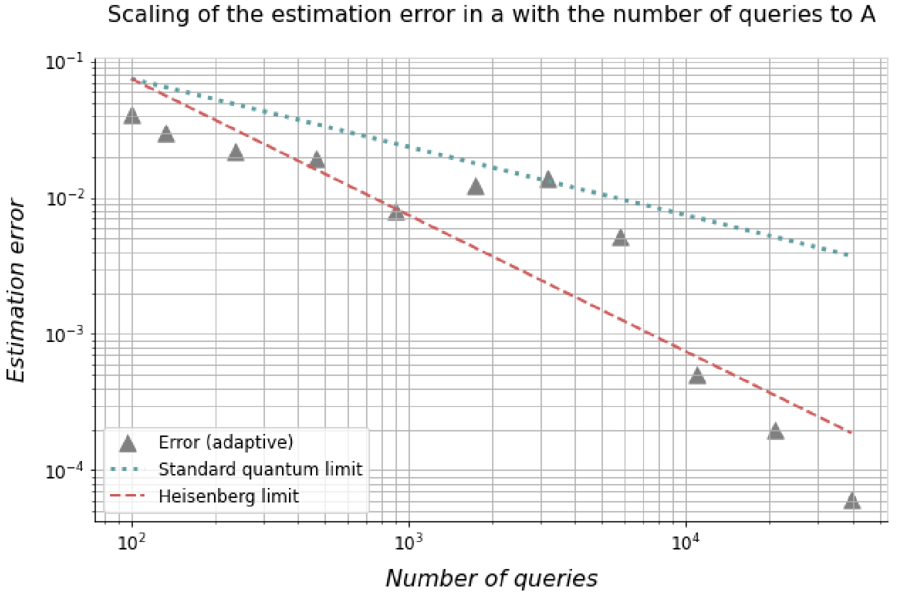
Bayesian amplitude estimation

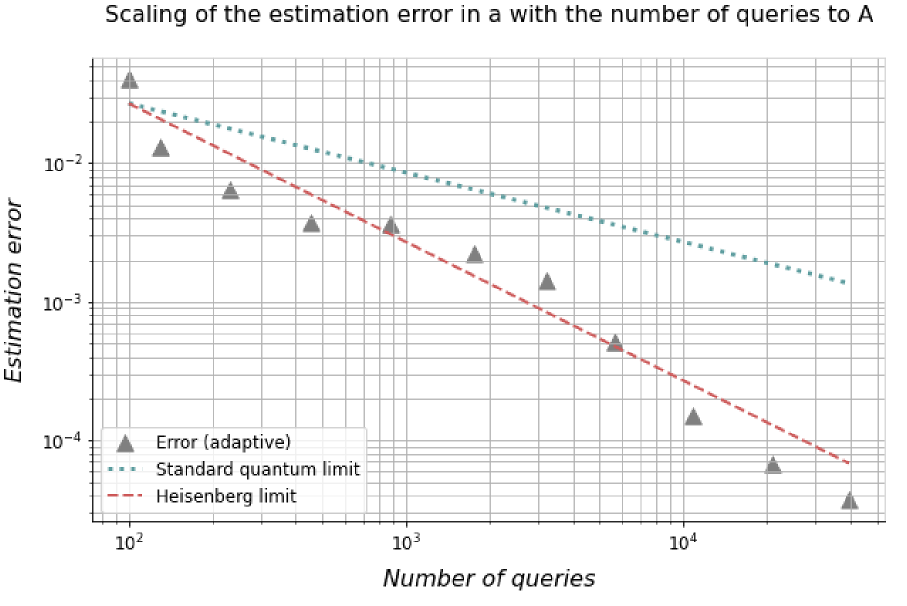
Bayesian inference offers a natural paradigm for solving this problem. Unlike less general protocols, it inherently offers:

$$P(\theta \mid D) = \frac{L(\theta \mid D; E)P(\theta)}{P(D; E)}$$

- The possibility of incorporating **noise models**;
- **Flexibility** to tailor the protocol;
- Various ways of negotiating the **trade-offs**

Bayesian amplitude estimation





mean
results

median
results

$a = 1$

Bayesian amplitude estimation

We can make the most of the open-endedness of Bayesian inference by:

- Properly handling **noise**;
- Using the **flexibility** to our benefit;
- Seeking the best **cost-to-benefit** ratio (across several types of *benefit* – quantum enhancement, noise resilience – and *cost* – classical processing, online processing, optimization, discretization, quantum resources, quantum depth, overheads).

Robust amplitude estimation

- Bayesian inference with engineered likelihood functions
- Greedy variance reduction (adaptively minimize proxies thereof)
- Work under a Gaussian assumption (making the representation analytically tractable
- Simple noise model

By Wang et al, 2021.

Conclusion

- All **QAE algorithms suffer with** the presence of **noise**, but they **are affected differently**: strategies with better idealized behavior may actually do worse once noise is introduced.
- We can expect that different **algorithms respond differently to different types of noise**, both qualitatively and quantitatively.
- Often, **algorithm design focuses on "hardware-friendly" circuits**. To make the most out of the available resources, the **processing should be "hardware-friendly"** as well.

Thank you for your attention!

Questions?